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Routh-Hurwitz Stability Criteria

Control Theory

Abstract
The Routh-Hurwitz stability criterion is a fundamental mathematical tool used in control system analysis to determine the stability of linear time-invariant (LTI) systems. This criterion provides a systematic method to assess whether all roots of a given characteristic equation lie in the left-half of the complex plane, ensuring system stability without explicitly computing the roots. By constructing the Routh array, stability conditions are derived based on the number of sign changes in the first column of the array. This method is widely applied in control engineering, electrical systems, and signal processing to evaluate system behavior and design stable controllers. The Routh-Hurwitz criterion offers computational efficiency and deep insight into system dynamics, making it an essential tool in control theory.

Keywords
Routh-Hurwitz criterion — Stability analysis — Control systems — Linear time-invariant (LTI) systems

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Routh-Hurwitz Stability Criterion

The Routh-Hurwitz Stability Criterion is a mathematical technique used to determine whether the roots of a polynomial lie on the left-hand side of the s-plane. It is a valuable tool as it allows one to assess the stability of a system without finding the roots of the characteristic equation.

1. Conditions for stability

There are certain necessary and sufficient conditions to determine the stability of a system using the Routh-Hurwitz criterion. These are outlined below:

1.1 Necessary condition

All the coefficients of the characteristic polynomial must be positive. If any coefficient is zero or negative, the system is potentially unstable.

$$1 + K(s)G(s)H(s) = \Delta(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0 \quad (1)$$

However, in cases where all coefficients are indeed positive, it does not guarantee the system's stability. There may still exist roots in the right half of the complex plane or on the imaginary axis.

1.2 Necessary and sufficient condition using Routh table

The necessary and sufficient condition for establishing stability using Routh-Hurwitz criteria is that every element must have a positive value or have no sign change. When this condition is not met, the system is unstable. Each sign change indicates a pair of roots in the right half-plane, suggesting instability.

2. Routh Table

In the construction of the Routh array, the coefficients of the characteristic equation are organized into two rows. This arrangement starts with the coefficients of s^n and s^{n-1} in that order, and then continues with the even-numbered and odd-numbered coefficients, as illustrated below:

$$\Delta(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0 \quad (2)$$

s^n	a_0	a_2	a_4	a_6	$\dots$
s^{n-1}	a_1	a_3	a_5	a_7	$\dots$

The following rows are added subsequently to complete the Routh array:

s^{n-2}	$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$	$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$	$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$	$\dots$
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1}$	$\dots$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
s^0	a_n			

This process continues until there are no more c_i elements remaining. Subsequently, the remaining rows are constructed in a similar manner, extending down to the s^0-row. The entire array takes on a triangular shape, with the noteworthy observation that both the s^1-row and the s^0-row consist of just a single term. It's important to emphasize that when constructing the Routh array, any absent terms are treated as zeroes. Additionally, throughout the process, we have the flexibility to multiply or divide all the elements in any row by a positive constant, a step taken to simplify the computational effort. This adjustment, however, does not alter the signs of the elements in the first column.

3. Routh-Hurwitz examples

Some basic examples are shown :

3.1 Stable system

Let's consider a fourth-order system with the characteristic equation:

$$s^4 + 10s^3 + 18s^2 + 16s + 5 = 0 \tag{3}$$

s^4	1	18	5
s^3	~~10~~5	~~16~~8	
s^2	$\frac{5 \cdot 18 - 1 \cdot 8}{5} = \frac{82}{5}$	$\frac{5 \cdot 5 - 1 \cdot 0}{5} = 5$	
s^1	$\frac{(82/5) \cdot 16 - 10 \cdot 5}{(82/5)} = \frac{531}{82}$		
s^0	$\frac{(531/41) \cdot 5 - (82/5) \cdot 0}{(531/41)} = 5$		

The elements of the first column of the table mentioned above are all positive, and hence the system is stable.

3.2 Unstable system

$$3s^4 + 20s^3 + 5s^2 + 5s + 2 = 0 \tag{4}$$

s^4	3	5	2
s^3	~~20~~4	~~5~~1	
s^2	$\frac{4 \cdot 5 - 3 \cdot 1}{4} = \frac{17}{4}$	$\frac{4 \cdot 2 - 3 \cdot 0}{4} = 2$	
s^1	$\frac{(17/4) \cdot 1 - 4 \cdot 2}{17/4} = -\frac{15}{17}$		
s^0	2		

Analyzing the first column of the Routh array reveals two sign changes, one from $17/4$ to $-15/17$ and another from $-15/17$ to 2. Consequently, the system being examined is unstable, featuring two poles situated in the right half of the s-plane. It's important to note that the Routh stability criterion exclusively provides the count of roots within the right half of the s-plane. This method does not furnish details about the actual root values or differentiate between real and complex roots.

4. Special cases of Routh-Hurwitz criteria

The above procedures fail in some cases listed below :

- All elements of a row are zero
- The first element of a row is zero

4.1 All elements of a row are zero

When a row with all zero elements is encountered, the subsequent row becomes undefined, halting the Routh array formation prematurely.

An all-zero row signifies that the given characteristic polynomial includes an even polynomial as a factor. An even polynomial is characterized by having terms where the exponents of "s" are either even integers or zero. This even polynomial factor is called the auxiliary polynomial.

Formation of Auxiliary Polynomial - Let's call the row just above the all-zero polynomial, the auxiliary row. The elements of this auxiliary row are the coefficients of the auxiliary polynomial $A(s)$. The order of the polynomial is the exponent of this critical row. Once $A(s)$ is found, the stability may be analyzed in either of the two ways:

- $A(s)$ is differentiated with respect to "s" and the all-zero row is replaced with the coefficients of $dA(s)/ds$. The construction of the array then continues as usual.
- The roots of $A(s)$ are also the roots of the given characteristic equation, which must be tested separately (Using the Routh-Hurwitz Criterion or other methods). The characteristic equation is divided by $A(s)$ to find the other factor, $B(s)$. Now, the Routh-Hurwitz criterion is applied to $B(s)$.

4.1.1 Example 1

Let's consider a sixth order characteristic equation:

$$\Delta(s) = s^6 + 2s^5 + 13s^4 + 24s^3 + 39s^2 + 54s + 27 \tag{5}$$

s^6	1	13	39	27
s^5	2	24	54	
s^4	1	12	27	
s^3	0	0		

The auxiliary polynomial $A(s)$ can be formed using the s^4 row, shown below:

$$A(s) = s^4 + 12s^2 + 27 \tag{6}$$

The derivative of $A(s)$ with respect to "s" is :

$$\frac{dA(s)}{d(s)} = 4s^3 + 24s \tag{7}$$

The zeros in the s^3-row are now replaced by the coefficients 4 and 24. The Routh array then becomes :

s^6	1	13	39	27
s^5	2	24	54	
s^4	1	12	27	
s^3	4	24		
s^2	6	27		
s^1	6			
s^1	27			

Since there are no sign changes in the first column of the routh array, the polynomial $\Delta(s)$ does not have any root in the right half of the s-plane. However, we know that the roots of $A(s)$ are the roots of $\Delta(s)$, so let's see the roots of $A(s)$:

$$A(s) = s^4 + 12s^2 + 27 = 0 \tag{8}$$

The roots of $A(s)$ are $s = +j\sqrt{3}, -j\sqrt{3}, +j3, -j3$, which are also the roots of $\Delta(s)$. Since they lie on the imaginary axis, the system is marginally stable and will oscillate.

4.1.2 Example 2

Lets take the following example :

$$\Delta(s) = s^5 + 2s^4 + 6s^3 + 26s^2 + 5s + 80 = 0 \tag{9}$$

s^1-row is all zero row, so s^2-row becomes the auxiliary row.

s^5	1	6	5
s^4	~~2~~1	~~26~~13	~~80~~40
s^3	~~−7~~ −1	~~−35~~ −5	
s^2	~~8~~1	~~40~~5	
s^1	0		

So, the auxiliary equation $A(s)$ is :

$$A(s) = s^2 + 5 \tag{10}$$

Roots of the $A(s) = 0$ are $s = +j\sqrt{5}$ and $s = -j\sqrt{5}$. $A(s)$ is one of the factors of the characteristic equation. To find the other factor, $\Delta(s)$ should be divided by $A(s)$.

$$\frac{\Delta(s)}{A(s)} = s^3 + 2s^2 + s + 16 \tag{11}$$

For the system to be stable, the roots of the above equation should not lie on the right half plane. That can be verified using Routh-Hurwitz criteria: Two sign changes in the first

s^3	1	1
s^2	~~2~~1	~~16~~8
s^1	−7	
s^0	8	

column indicate the presence of two roots in the right-half plane. Therefore, the system is unstable.

4.2 The first element of a row is zero

If the first element of a row is zero, the terms in the subsequent row become infinite, making it impossible to continue forming the array. To solve this problem, we replace the zero with a small positive number ε and construct the array. The limit $\varepsilon \to 0$ is then applied to identify changes in signs in the first column, providing insights into the number of roots in the right half-plane.

4.2.1 Example 1

Let's consider a fifth-order polynomial :

$$s^5 + 3s^4 + 2s^3 + 6s^2 + 8s + 12 = 0 \tag{12}$$

The first element in the s^2-row is $(2\varepsilon - 4)/\varepsilon$, which has a

s^5	1	2	8
s^4	~~3~~1	~~6~~2	~~12~~4
s^3	~~0~~ε	4	
s^2	$\frac{2\varepsilon - 4}{\varepsilon}$	4	
s^1	$4 - \frac{4\varepsilon^2}{2\varepsilon - 4}$		
s^0	4		

negative sign as $\varepsilon \to 0$. Here, the magnitude can be ignored. The first term of s^1-row is $(4 - 4\varepsilon^2/(2\varepsilon - 4))$, which has a limiting value of +4 as $\varepsilon \to 0$. If the first column is analyzed again, it can be observed that there are two sign changes. The first sign change is from s^3 to s^2, and the second is from s^2 to s^1. This suggests that two poles are in the right half plane (the system is unstable).

4.2.2 Example 2

Let's consider a sixth-order polynomial :

$$\Delta(s) = s^6 + s^5 + 5s^4 + 5s^3 + 7s^2 + 6s + 2 \tag{13}$$

s^6	1	5	7	2
s^5	1	5	6	
s^4	$\not{0}\varepsilon$	1	2	
s^3	$\frac{5\varepsilon-1}{\varepsilon}$	$\frac{6\varepsilon-2}{\varepsilon}$		
s^2	$1-\frac{6\varepsilon-2}{5\varepsilon-1}\varepsilon$	2		
s^1	$\frac{6\varepsilon-2}{\varepsilon}-\frac{2\left(\frac{5\varepsilon-1}{\varepsilon}\right)}{1-\left(\frac{6\varepsilon-2}{5\varepsilon-1}\right)\varepsilon}\simeq 0$			

As $\varepsilon \to 0$, the elements of the s^1-row approach zero, suggesting the presence of roots on the imaginary axis in the s-plane. To confirm this, the auxiliary polynomial must be analyzed. If no imaginary-axis roots are present, the zero row is replaced with the coefficients of the derivative of the auxiliary polynomial. However, if imaginary-axis roots are identified, the original polynomial is divided by the auxiliary polynomial, and the stability test is conducted on the resulting remainder polynomial. In this example, the auxiliary polynomial is (let $\varepsilon \to 0$ in s^2-row):

$$A(s) = s^2 + 2 = 0 \tag{14}$$

The above auxiliary equation has two roots, $+j\sqrt{2}\omega$ and $-j\sqrt{2}\omega$, on the imaginary axis. Dividing the original 6^{th}-order polynomial $\Delta(s)$ with $A(s)$, we get:

$$\frac{\Delta(s)}{A(s)} = s^4 + s^3 + 3s^2 + 3s + 1 \tag{15}$$

The Routh array for this polynomial is : As $\varepsilon \to 0+$, there

s^4	1	3	1
s^3	1	3	
s^2	$\not{0}\varepsilon$	1	
s^1	$\frac{3\varepsilon-1}{\varepsilon}$		
s^0	1		

are two sign changes in the first column of the array. This indicates that there are two roots in the right half of the s-plane. The original polynomial $\Delta(s)$, therefore, has two roots in the right-half plane and two roots on the imaginary axis.

5. Advantages of Routh-Hurwitz criteria

- The system's stability can be determined without finding the roots of the characteristic equation.
- The relative stability of the system can be determined easily.
- This method also allows one to determine where the root locus intersects the imaginary axis.

6. Limitations of Routh-Hurwitz Criterion

- This is only applicable to linear systems.
- It does not compute the exact location of the poles on either the right or left half of the S-plane.

References

[1] Norman S. Nise. 2000. Control Systems Engineering (3rd. ed.). John Wiley & Sons, Inc., USA.